EXPONENTIAL ESTIMATES AND THE SADDLE POINT PROPERTY FOR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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M. A. Cruz and J. K. Hale

1. Introduction. If A_j, B_j, $j = 0,1,\dots,N$, are $n \times n$ constant matrices, $\det A_0 \neq 0$, and $0 = \omega_0 < \omega_1 < \cdots < \omega_N = r$ are real numbers, then a differential-difference equation of neutral type is

$$(1.1) \qquad \sum_{j=0}^{N} A_j \dot{x}(t-\omega_j) = \sum_{j=0}^{N} B_j x(t-\omega_j).$$

A fundamental problem is to determine in what sense the asymptotic behavior of the solutions of (1.1) is given from a knowledge of the solutions of the characteristic equation

$$(1.2) \qquad \det \Delta(\lambda) = 0, \quad \Delta(\lambda) = \sum_{j=0}^{N} (\lambda A_j - B_j) e^{-\lambda\omega_j}.$$

Without exception, the results in the literature (see [1-5]) are based on the assumption that the initial function φ and its derivative are defined. The estimate for the growth of the solution and not the derivative of the solution is then expressed in terms of the roots of (1.2) and $\varphi, \dot{\varphi}$. This is very unsatisfactory for the following reason. If a well-posed initial value problem has been formulated for (1.1), then one has chosen a space S of functions mapping $[-r,0]$ into E^n such that for any initial function φ in S there is a solution $x(\varphi)$ of (1.1) with initial value φ which is continuous in φ and the restriction of $x(\varphi)$ to $[t-r,t]$ always belongs to S. This defines a mapping $T(t): S \to S$ and one would hope that the norm of this linear mapping could be obtained from the solutions of (1.2). On the other hand, the results in [1-5] use more smoothness properties for φ than are obtained for $x(\varphi)$ and, therefore, one is not estimating the norm of $T(t)$. It is the main purpose of this paper to give a class of equations (1.1) for which one can estimate the norm of $T(t)$ using (1.2). The results are stated in terms of general functional differential equations which include differential-difference equations. An application to perturbed linear equations is indicated by discussing the saddle point property for nonlinear autonomous systems.

Finally, to avoid unnecessary complications in the specification of the basic space S, we use the approach in [5] by considering the integrated form of (1.1),

$$\frac{d}{dt}\left[\sum_{k=0}^{N} A_k x(t-\omega_k)\right] = \sum_{k=0}^{N} B_k x(t-\omega_k). \tag{1.3}$$

For this equation, one has a well-posed initial value problem for any initial function φ which is continuous on $[-r,0]$ since it is not required that x be differentiable in t, but only that $\sum_{k=0}^{N} A_k x(t-\omega_k)$ be differentiable. Consequently, it is possible to choose S as the space of continuous functions.

2. Notations and summary of known results. Let $R^{+} = [0,\infty)$, E^n be

a real or complex n-dimensional linear vector space with norm $|\cdot|$, $r \geq 0$ a given real number, and C be the space of continuous functions mapping $[-r,0]$ into E^n with $|\varphi| = \sup_{-r\leq\theta\leq 0}|\varphi(\theta)|$. Single bars are generally used to denote norms in different spaces, but no confusion should arise. If x is a continuous function defined on $[\sigma-r,\sigma+A]$, $A \geq 0$, then, for each $t \in [\sigma,\sigma+A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. Suppose

$$(2.1)\qquad \begin{aligned} &\text{a)}\quad L(\varphi) = \int_{-r}^{0}[d\eta(\theta)]\varphi(\theta) \\ &\text{b)}\quad g(\varphi) = \int_{-r}^{0}[d\mu(\theta)]\varphi(\theta) \\ &\text{c)}\quad \left|\int_{-s}^{0}[d\mu(\theta)]\varphi(\theta)\right| \leq \gamma(s)|\varphi| \\ &\text{d)}\quad D(\varphi) = \varphi(0) - g(\varphi) \end{aligned}$$

where η,μ are $n \times n$ matrix functions with elements of bounded variation on $[-r,0]$ and $\gamma(s)$, $s \geq 0$, is continuous with $\gamma(0) = 0$. An autonomous linear functional differential equation is defined to be

$$(2.2)\qquad \frac{d}{dt}D(x_t) = L(x_t).$$

A solution $x = x(\varphi)$ of (2.2) through $(0,\varphi)$, $\varphi \in C$, is a continuous function defined on an interval $[-r,A]$, $A > 0$, such that $x_0 = \varphi$ and $D(x_t)$ is continuously differentiable for $t \in (0,A)$

and satisfies (2.2). It is proved in [5] that there is a unique solution $x(\varphi)$ through $(0,\varphi)$ defined on $(-\infty,\infty)$ and $x(\varphi)(t)$ is continuous in t,φ. If the transformation $T(t): C \to C$ is defined by

$$x_t(\varphi) \overset{\text{def}}{=} T(t)\varphi \tag{2.3}$$

then it is also shown in [5] that $\{T(t),\ t \geq 0\}$ is a strongly continuous semigroup of linear operators with infinitesimal generator $A: \mathscr{D}(A) \to C$, $A\varphi(\theta) = \dot{\varphi}(\theta)$,

$$\mathscr{D}(A) = \{\varphi \in C: \dot{\varphi} \in C,\ \dot{\varphi}(0) = g(\dot{\varphi}) + L(\varphi)\} \tag{2.4}$$

and the spectrum $\sigma(A)$ consists of all those λ for which

$$\det \Delta(\lambda) = 0,\ \Delta(\lambda) = \lambda\left[I - \int_{-r}^{0} e^{\lambda\theta} d\mu(\theta)\right] - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta). \tag{2.5}$$

Moreover, there are real constants $K \geq 1$, a such that

$$|x_t(\varphi)| = |T(t)\varphi| \leq Ke^{at}|\varphi|,\quad t \geq 0,\ \varphi \in C. \tag{2.6}$$

The basic problem is now to determine the relationship between $\inf\{a:$ there exists a $K = K(a)$ so that (2.6) holds$\}$ and $\sup\{\operatorname{Re}\lambda: \lambda$ satisfies (2.5)$\}$. For any λ satisfying (2.5), there is a solution $e^{\lambda t}b$ of (2.2) for some vector b. Therefore,

$\sup\{\lambda: \cdots\} \leq \inf\{a: \cdots\}$. It certainly seems as if these two numbers should be the same, but we are unable to prove this at the present time. In [6], D. Henry has shown these numbers are equal if the space C is replaced by $W_2^{(1)}$, the space of functions which have square integrable first derivatives. In order to obtain some results in C, we impose in the next section some conditions on the "difference operator" D.

3. The characteristic equation. Suppose μ° is an $n \times n$ matrix function whose elements are of bounded variation, $\gamma^\circ(\delta)$ is a continuous nonnegative scalar function defined on $[0,\infty)$, $\gamma^\circ(0) = 0$, and let

$$\text{(3.1)} \qquad \begin{aligned} &\text{a)} \quad D^\circ(\varphi) = \varphi(0) - g^\circ(\varphi) \\ &\text{b)} \quad g^\circ(\varphi) = \int_{-r}^{0} [d\mu^\circ(\theta)]\varphi(\theta) \\ &\text{c)} \quad \left|\int_{-s}^{0} [d\mu^\circ(\theta)]\varphi(\theta)\right| \leq \gamma^\circ(s)\sup_{-s\leq\theta\leq 0}|\varphi(\theta)|, \quad 0 \leq s \leq r. \end{aligned}$$

In this section, we consider in detail a special case of (2.2); namely, the functional "difference" equation

$$\text{(3.2)} \qquad \begin{aligned} D^\circ(y_t) &= D^\circ(\varphi), \quad t \geq 0, \\ y_0 &= \varphi \end{aligned}$$

and, in particular, the nature of the characteristic equation of this system. Afterwards, the results will be applied to obtain information about the characteristic equation of the more general system (2.2).

Let us denote the semigroup and infinitesimal generator associated with (3.1) by $T^{o}(t)$ and A^{o}, respectively, and let

$$\Delta^{o}(\lambda) = I - \int_{-r}^{0} e^{\lambda\theta} d\mu^{o}(\theta). \tag{3.3}$$

The characteristic matrix of (3.2) is then given by $\lambda\Delta^{o}(\lambda)$.

Along with system (3.2), we consider the "homogeneous" equation

$$\begin{aligned} D^{o}(y_t) &= 0, \qquad t \geq 0 \\ y_0 &= \varphi, \qquad D^{o}(\varphi) = 0. \end{aligned} \tag{3.4}$$

Definition 3.1. If D^{o} is given in (3.1), the order $a_{D^{o}}$ of D^{o} is defined by

$$a_{D^{o}} = \inf\{\text{real } a: \text{ there is a } K(a) \text{ with } |T^{o}(t)\varphi| \leq K(a)e^{at}|\varphi|,\ t \geq 0, \text{ for all } \varphi \text{ with } D^{o}(\varphi) = 0\}. \tag{3.5}$$

This definition is equivalent to

$$a_{D^{o}} = \inf\{\text{real } a: \text{ for any } \varphi \text{ in } C,\ D^{o}(\varphi) = 0, \text{ there is a } K(\varphi,a) \text{ with } |T^{o}(t)\varphi| \leq K(\varphi,a)e^{at},\ t \geq 0\}. \tag{3.6}$$

In fact, since D° is continuous and linear, the set consisting of all φ in C such that $D^{\circ}(\varphi) = 0$ is a Banach space and the operator $T^{\circ}(t)$ is a continuous linear mapping of this space into itself for each $t \geq 0$. The principle of uniform boundedness now implies that the set on the right hand side of (3.6) belongs to the set on the right hand side of (3.5). The converse inclusion is obvious and this shows that $a_{D^{\circ}}$ may be defined by either (3.5) or (3.6).

Notice that $a_{D^{\circ}}$ is determined by the exponential behavior of the solutions of the homogeneous equation (3.4) and not the complete equation (3.2). The reason for this is the following: every constant function satisfies (3.2) regardless of the nature of the operator D°. This is a consequence of the fact that $\lambda = 0$ always satisfies the characteristic equation. The homogeneous equation is considered to eliminate this obvious common relationship among all operators D°.

In general, we do not know how to relate the number $a_{D^{\circ}}$ with the roots of the characteristic equation. However, the following lemma is a special case for which this relationship is known. A more general result is contained in [7].

Lemma 3.1. If

$$(3.7) \qquad D^{\circ}(\varphi) = \varphi(0) - \sum_{k=1}^{N} A_k \varphi(-\tau_k), \quad 0 < \tau_k \leq r,$$

where τ_j/τ_k is rational if $N > 1$, then

$$(3.8) \qquad a_{D^o} = \sup \{\operatorname{Re} \lambda : \det \left(I - \sum_{k=1}^{N} A_k e^{-\lambda\tau_k}\right) = 0\}.$$

Proof: If $D^o(\varphi) = \varphi(0)$, then $a_{D^o} = -\infty$. Suppose b_{D^o} is the sup in (3.8) and $a > b_{D^o}$. If y is a solution of $D^o y_t = 0$, $x_o = \varphi$, and $y(t) = e^{at}z(t)$, then

$$D^o(e^{a\cdot}z_t) = 0$$

$$z_0 = e^{-a\cdot}\varphi.$$

If we let $D_1(\psi) = D^o(e^{a\cdot}\psi)$, then

$$D_1(\psi) = \psi(0) - \sum_{k=1}^{N} A_k e^{-a\tau_k}\psi(-\tau_k)$$

and

$$\begin{aligned} b_{D_1} &= \sup \{\operatorname{Re} \nu : \det \left(I - \sum_{k=1}^{N} A_k e^{-(\nu+a)\tau_k}\right) = 0\} \\ &= \sup \{\operatorname{Re} (\lambda - a) : \det \left(I - \sum_{k=1}^{N} A_k e^{-\lambda\tau_k}\right) = 0\} \\ &= a_{D^o} - a < 0. \end{aligned}$$

Therefore, D_1 is a uniformly stable operator and Lemma 3.2 in [8] implies the existence of an $\alpha > 0$, $\beta > 0$, $\beta_1 > 0$, such that

$$|z_t(e^{-\bar{a}\cdot}\varphi)| \leq \beta e^{-\alpha t}|e^{-a\cdot}\varphi| \leq \beta_1 e^{-\alpha t}|\varphi|, \qquad t \geq 0.$$

Consequently, there is a $\beta_2 > 0$ such that

$$|y_t| \leq \beta_2 e^{(a-\alpha)t}|\varphi| \leq \beta_2 e^{at}|\varphi|, \qquad t \geq 0.$$

This implies $a_{D^o} \leq b_{D^o}$.

For any $\varepsilon > 0$, there is a λ with $b_{D^o} - \varepsilon < \mathrm{Re}\,\lambda \leq b_{D^o}$ and an n-vector c such that $y(t) = e^{\lambda t}c$ is a solution of $D^o y_t = 0$. Therefore, $a_{D^o} > b_{D^o} - \varepsilon$ for every $\varepsilon > 0$. This proves $a_{D^o} = b_{D^o}$ and the lemma.

<u>Lemma 3.2.</u> There exist φ_j in $\mathscr{D}(A^o)$, $j = 1,2,\ldots,n$, such that if $\Phi = (\varphi_1,\ldots,\varphi_n)$, then $D^o(T^o(t)\Phi) = D^o(\Phi) = I$, the identity. Also, for any $a > a_{D^o}$, there is an $M = M(a)$ such that

$$|T^o(t)\Phi| \leq M(1+e^{at}), \qquad t \geq 0. \tag{3.9}$$

<u>Proof</u>: Let us consider the equation (3.4) and, in particular, all solutions of this equation which are polynomials in t. If we let

$$P^o_{j+1}(\lambda) = \frac{1}{j!}\frac{d^j}{d\lambda^j}\Delta^o(\lambda), \quad j = 0,1,2,\ldots$$

where $\Delta^{o}(\lambda)$ is defined in (3.3), then a direct calculation shows that

$$(3.10) \qquad y(t) = \sum_{k=0}^{m} \alpha_{m-k} \frac{t^k}{k!}$$

is a solution of equation (3.4) if and only if

$$(3.11) \qquad A_m^o \alpha^m = 0,$$

$$A_m^o = \begin{bmatrix} P_1^o(0) & P_2^o(0) & \cdots & P_{m+1}^o(0) \\ 0 & P_1^o(0) & \cdots & P_m^o(0) \\ \vdots & & & \\ 0 & 0 & \cdots & P_1^o(0) \end{bmatrix}, \quad \alpha^m = \begin{bmatrix} \alpha_m \\ \alpha_{m-1} \\ \\ \alpha_0 \end{bmatrix}$$

Let m_0 be the smallest integer such that the equation (3.4) has no polynomial solution of degree m. Then for every $\alpha_0 \neq 0$, and every $\alpha_1, \ldots, \alpha_{m_0}$, the equation $A_{m_0}^o \alpha^{m_0} \neq 0$. If we choose the vector α^{m_0-1} such that $A_{m_0-1}^o \alpha^{m_0-1} = 0$, then the equation (3.11) is equivalent to the equation

$$P_1^o(0)\alpha_{m_0} + P_2^o(0)\alpha_{m_0-1} + \cdots + P_{m_0+1}^o(0)\alpha_0 = 0.$$

On the other hand, by the choice of m, this equation has no solution except $\alpha^{m_0} = 0$ if $\alpha_0 \neq 0$. Therefore, the equation

$$P_1^o(0)\alpha_{m_0} + P_2^o(0)\alpha_{m_0-1} + \cdots + P_{m_0+1}^o(0)\alpha_0 = b$$

has a unique solution α^{m_0} with $\alpha_0 \neq 0$ for every b. In particular, the matrix equation

$$A_{m_0}\alpha^{m_0} = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where I is the $n \times n$ identity matrix, has a unique $nm_0 \times m_0$ matrix solution which we denote by (3.11) with each α_j an $n \times n$ matrix.

If y is defined by (3.10) for this α^{m_0} and $m = m_0$, we see that

$$\begin{aligned} D^o(y_t) &= \sum_{k=0}^{m_0} D^o\left((t+\cdot)^k \frac{\alpha_{m_0-k}}{k!}\right) \\ &= \sum_{k=0}^{m_0}\left[\sum_{j=0}^{k} P^o_{j+1}(0)\frac{t^{k-j}}{(k-j)!}\right]\alpha_{m_0-k} \\ &= \sum_{\ell=0}^{m_0}\left[\sum_{\nu=0}^{m_0-\ell} P^o_{\nu+1}(0)\alpha_{m_0-\ell-\nu}\right]\frac{t^\ell}{\ell!} \\ &= I, \quad t \in (-\infty,\infty). \end{aligned}$$

Therefore, $y(t)$ is a continuously differentiable solution of (3.2) on $(-\infty,\infty)$ with initial value Φ at $t = 0$ such that $D^o\Phi = I$. Since $D^o(\dot{y}_t) = 0$ for t in $(-\infty,\infty)$, it follows that Φ is in $\mathscr{D}(A^o)$.

It remains only to prove the estimate (3.9). For any $a > a_{D^o}$, there is a constant $M_1 = M_1(a)$ such that for any $\bar{a}$ with $a_{D^o} + (a-a_{D^o})/2 < \bar{a} < a$,

$$|\dot{y}_t| \le M_1 e^{\bar{a}t}, \qquad t \ge 0,$$

since $\dot{y}_t$ satisfies (3.4). Choose $\bar{a} \neq 0$. Since

$$y(t+\theta) = \Phi(0) + \int_0^{t+\theta} \dot{y}(s)ds$$

for $t \geq 0$, $-r \leq \theta \leq 0$, this yields the estimate

$$|y(t)| \leq M_2\left(1 + \frac{e^{\bar{a}t}}{\bar{a}}\right), \quad t \geq 0.$$

Since $\bar{a} < a$, one can obtain the estimate (3.9).

For any $H \in C([0,\infty),E^n)$, $H(0) = 0$, it follows from [8] that there is an $n \times n$ matrix function $B^0: [-r,\infty) \to E^{n^2}$ of bounded variation on compact sets of $[-r,\infty)$, $B^0(t) = 0$, $-r \leq t \leq 0$, such that the solution of

$$\begin{aligned} D^0(y_t) &= D^0(\varphi) + H(t), \quad t \geq 0, \\ y_0 &= \varphi \end{aligned} \tag{3.12}$$

is given by the variation of constants formula as

$$y_t = T^0(t)\varphi - \int_0^t [d_s B^0_{t-s}]H(s). \tag{3.13}$$

Lemma 3.3. For any $a > a_{D^0}$, $\varepsilon > 0$, $a + \varepsilon \neq 0$ there is an $M = M(a\ \varepsilon) > 0$ such that

$$\left|\int_0^t [d_s B^0_{t-s}]H(s)\right| \leq M(1+e^{at})e^{\varepsilon t}\sup_{0\leq s\leq t}|H(s)|, \quad t \geq 0. \tag{3.14}$$

Proof: If y is the solution of (3.2) and Φ is given in Lemma 3.2, then $z_t = y_t - T^0(t)\Phi D^0(\varphi)$ satisfies $D^0(z_t) = 0$, $z_0 = \varphi - \Phi D^0(\varphi)$.

Therefore, for any $a > a_{D^o}$, there is a K_1 such that

$$|z_t| \le K_1 e^{at}|z_0| = K_1 e^{at}|\varphi - \Phi D^o(\varphi)|.$$

Lemma 3.2 and the continuity of D^o imply the existence of a $K_2 = K_2(a)$ such that

$$|T^o(t)\varphi| \le K_2(1+e^{at})|\varphi|, \qquad t \ge 0.$$

Using an argument similar to the proof of Theorem 3.1 in [9], there is a $K = K(a) > 0$ such that

$$|B^o(t)| + \int_0^1 |d_s B^o(t-s)| \le K(1+e^{at}), \qquad t \ge 0.$$

If $k = k(t)$ is the integer such that $k \le t < k + 1$, then, for any $\varepsilon > 0$, $a + \varepsilon \ne 0$,

$$\begin{aligned}
\left|\int_0^t [d_s B^o_{t-s}]H(s)\right| &\le K \sum_{j=1}^{k+1} (1+e^{aj}) \sup_{0\le s\le t}|H(s)| \\
&\le \left[K(k+1) + \sum_{j=1}^{k+1} e^{(a+\varepsilon)j}\right] \sup_{0\le s\le t}|H(s)| \\
&\le \left[K(t+1) + \frac{e^{(a+\varepsilon).(k+1)} - 1}{e^{a+\varepsilon} - 1}\right] \sup_{0\le s\le t}|H(s)| \\
&\le M(1+e^{at})e^{\varepsilon t} \sup_{0\le s\le t}|H(s)|
\end{aligned}$$

for some constant M. This proves the lemma.

Lemma 3.4. For any $a > a_{D^o}$, the roots of

$$\det \Delta^o(\lambda) = 0, \quad \Delta^o(\lambda) = I - \int_{-r}^{o} e^{\lambda\theta} d\mu^o(\theta) \tag{3.15}$$

have real parts less than or equal to a and there is a $\delta(a) > 0$ such that $|\det \Delta^o(\lambda)| \geq \delta(a)$ on $\operatorname{Re} \lambda = a$.

Proof: If λ satisfies (3.15), then there is a nonzero n-vector b such that $y(t) = e^{\lambda t} b$ satisfies $D^o(y_t) = 0$. Definition (5.1) of a_{D^o} implies the first part of the lemma.

If the second statement of the lemma is not true, there is a sequence $\{\lambda_k\}$, $k = 1,2,\ldots$ of points on $\operatorname{Re} \lambda = a$ such that $|\det \Delta^o(\lambda_k)| \leq 1/k$, $k = 1,2,\ldots$. This implies the existence of an eigenvalue of $\Delta^o(\lambda_k)$ with modulus $\leq (1/k)^{1/n}$. Suppose ζ_k is such an eigenvalue of $\Delta^o(\lambda_k)$ and b_k, $|b_k| = 1$, is an eigenvector associated with ζ_k.

The function $y^k(t) = e^{\lambda_k t} b_k$ satisfies

$$D^o(y_t^k) = e^{\lambda_k t}\zeta_k b_k, \quad t \geq 0$$

$$y_0^k = e^{\lambda_k \cdot} b_k, \quad D^o(y_0^k) = \zeta_k b_k.$$

If Φ is the matrix defined in Lemma 3.2 and $z_t^k = y_t^k - T^o(t)\Phi\zeta_k b_k$

then

$$D^0(z_t^k) = (e^{\lambda_k t} - 1)\zeta_k b_k, \quad t \geq 0$$

$$z_0^k = x_0^k - \Phi\zeta_k b_k, \quad D^0(z_0^k) = 0.$$

The variation of constants formula (3.13) implies

$$z_t^k = T^0(t)z_0^k - \int_0^t [d_s B_{t-s}^0](e^{\lambda_k s} - 1)\zeta_k b_k.$$

From the fact that $D^0(z_0^k) = 0$, the definition of a_{D^0} and Lemmas 3.2 and 3.3, for any $\bar{a}$, $a_{D^0} < \bar{a} < a$, $\varepsilon > 0$, $\bar{a} + \varepsilon \neq 0$, there is a constant $M = M(\bar{a},\varepsilon)$ such that

$$(3.16) \qquad \begin{aligned} |y_t^k| &\leq |T^0(t)\Phi\zeta_k b_k| + |z_t^k| \\ &\leq M(1+e^{\bar{a}t})|\zeta_k| + Me^{\bar{a}t}[|\zeta_k| + \sup_{-r\leq\theta\leq 0} e^{a\theta}] \\ &\quad + M(1+e^{\bar{a}t})e^{\varepsilon t}|\zeta_k| \sup_{0\leq s\leq t}|e^{\lambda_k s} - 1|. \end{aligned}$$

On the other hand, the definition of y_t^k and the fact that $\bar{a} < a$ implies the existence of a $T > 0$ such that

$$|y_t^k| = e^{at}\sup_{-r\leq\theta\leq 0} e^{a\theta} > Me^{\bar{a}t}\sup_{-r\leq\theta\leq 0} e^{a\theta},$$

for $t \geq T$, $k = 1,2,\ldots$. Since $\zeta_k \to 0$ as $k \to \infty$, this contradicts

(3.16) and proves the lemma.

Lemma 3.5. Suppose D^o is defined in (3.1), $\Delta^o(\lambda)$ in (3.3), $\alpha \in \mathcal{L}^1([-r,0],E^{n^2})$, and η is an $n \times n$ matrix function of bounded variation. For any $a > a_{D^o}$, the equation

$$(3.17) \qquad \det \Delta(\lambda) = 0, \quad \Delta(\lambda) = \lambda[\Delta^o(\lambda) - \int_{-r}^{o} e^{\lambda\theta}\alpha(\theta)d\theta] - \int_{-r}^{o} e^{\lambda\theta}d\eta(\theta)$$

has only a finite number of roots λ with $\operatorname{Re} \lambda \geq a$.

Proof: If we consider $\Delta(\lambda)$ as the characteristic matrix of a neutral functional differential equation (2.2), then the estimate (2.6) implies there exists a real number c such that $\operatorname{Re} \lambda < c$ for all λ satisfying (3.17). If $a \geq c$, then the above lemma is true. If $a < c$, then Lemma 3.4 implies there is a $\delta = \delta(a,c) > 0$ such that $\det \Delta_0(\lambda) \geq \delta$, $a \leq \operatorname{Re} \lambda \leq c$. From (3.17), the Riemann-Lebesgue lemma, and the fact that μ^o satisfies (3.1c),

$$\det \Delta(\lambda) = \lambda^n \Delta^o(\lambda) + h(\lambda)$$

where $h(\lambda)/\lambda^n \to 0$ uniformly as $|\lambda| \to \infty$, $a \leq \operatorname{Re} \lambda \leq c$. Therefore, all zeros of (3.17) in this strip must be bounded. Since $\det \Delta(\lambda)$ is an entire function of λ, the lemma is proved.

4. Estimates on the complementary subspace. Suppose D^o is defined

in (3.1), $\alpha \in \mathcal{L}^1([-r,0],E^{n^2})$, η is an $n \times n$ matrix function whose elements are of bounded variation and let

$$(4.1) \qquad D(\varphi) = D^0(\varphi) - \int_{-r}^{0} \alpha(\theta)\varphi(\theta)d\theta \overset{\text{def}}{=} \varphi(0) - \int_{-r}^{0} [d\mu(\theta)]\varphi(\theta)$$

$$L(\varphi) = \int_{-r}^{0} [d\eta(\theta)]\varphi(\theta).$$

For the linear system (2.2) we denote the associated semigroup and infinitesimal generator by $T(t)$ and A, respectively. Recall that the spectrum $\sigma(A)$ of A coincides with the roots of the characteristic equation (2.5).

For any $a > a_{D^0}$, it follows from Lemma 3.5 that the equation (2.5) has only a finite number of roots λ with $\operatorname{Re}\lambda \geq a$. If $\Lambda_a = \{\lambda \in \sigma(A): \operatorname{Re}\lambda \geq a\}$, then it is shown in [5] that the space C can be decomposed by Λ_a as $C = P_a \oplus Q_a$ where P_a, Q_a are subspaces of C invariant under $T(t)$ and A, the space P_a is finite dimensional and corresponds to the initial values of all those solutions of (2.2) which are of the form $p(t)e^{\lambda t}$ where $p(t)$ is a polynomial in t and $\lambda \in \Lambda_a$. Therefore, the spectrum of A restricted to Q_a is $\sigma(A) \setminus \Lambda_a$. Our main goal in this section is to prove there is a constant $K(a)$ such that

$$|T(t)\varphi| \leq K(a)e^{at}|\varphi|, \quad t \geq 0, \quad \varphi \in Q_a.$$

To do this, we need the following lemma which is essentially contained in the proof of Theorem IV.1 of [5].

Lemma 4.1. Suppose a is a real number such that only a finite number of roots of (2.5) have real parts greater than or equal to a, there is a constant $m > 0$ such that, for all real ξ, $|\det(a+i\xi)| \geq m > 0$ and $\Delta^{-1}(a+i\xi) = \mathcal{O}(|\xi|^{-1})$ as $|\xi| \to \infty$. If C is decomposed by $\Lambda_a = \{\lambda \in \sigma(A): \operatorname{Re}\lambda \geq a\}$ as $C = P_a \oplus Q_a$, then there exists a $K = K(a) \geq 1$, such that

$$(4.2) \qquad |T(t)\varphi| \leq Ke^{at}(|\varphi|+|\dot{\varphi}|), \quad t \geq 0, \ \varphi \in \mathscr{D}(A) \cap Q_a.$$

For any $H \in C([0,\infty),E^n)$, $H(0) = 0$, it follows from [9] that there is an $n \times n$ matrix function $B: [-r,0] \to E^{n^2}$ of bounded variation on compact sets of $[-r,\infty)$, $B(t) = 0$, $-r \leq t \leq 0$, such that the solution of

$$(4.3) \qquad \frac{d}{dt}[D(x_t) - H(t)] = L(x_t), \quad t \geq 0,$$
$$x_0 = \varphi$$

is given by the variation of constants formula as

$$(4.4) \qquad x_t = T(t)\varphi - \int_0^t [d_s B_{t-s}]H(s) = T(t)\varphi + \int_0^t B_{t-s} d_s H(s).$$

If we let $x_t^{P_a}$ be the projection of x_t onto P_a defined by the above decomposition of C, then it follows that there is a $B_t^{P_a}$, $t \geq 0$, $B_0^{P_a} = 0$, of bounded variation on compact subsets of $[0,\infty)$ such that $x_t^{P_a}$ satisfies (4.4) with x_t, φ, B_t replaced by $x_t^{P_a}$, φ^{P_a}, $B_t^{P_a}$, respectively. If we define $B_t^{Q_a} = B_t - B_t^{P_a}$, then (4.4) is equivalent to

$$x_t^{P_a} = T(t)\varphi^{P_a} - \int_0^t [d_s B_{t-s}^{P_a}]H(s), \tag{4.5}$$
$$x_t^{Q_a} = T(t)\varphi^{Q_a} - \int_0^t [d_s B_{t-s}^{Q_a}]H(s).$$

<u>Theorem 4.1.</u> Suppose D is given in (4.1). If $a > a_{D^0}$ is such that $\lambda \in \sigma(A)$ implies $\operatorname{Re}\lambda \neq a$ and C is decomposed by $\Lambda_a = \{\lambda \in \sigma(A): \operatorname{Re}\lambda > a\}$ as $C = P_a \oplus Q_a$, then there is a constant $M = M(a)$ such that

$$|T(t)\varphi| \leq Me^{at}|\varphi|, \quad t \geq 0, \quad \varphi \in Q_a \tag{4.6}$$

$$|B_t^{Q_a}| + \int_0^1 |d_s B_{t-s}^{Q_a}| \leq Me^{at}, \qquad t \geq 0, \tag{4.7}$$

where B is the matrix occurring in the variation of constants formula (4.4) and $B_t^{Q_a}$ is defined as above.

<u>Proof:</u> <u>Case 1.</u> $\alpha = 0$, $L \equiv 0$; that is, the equation

$$D^0(x_t) = D^0(\varphi), \qquad t \geq 0, \quad x_0 = \varphi \in Q_{\Lambda_a}. \tag{4.8}$$

If Φ is the matrix given in Lemma 3.2, then

$$T^{o}(t)\varphi = T^{o}(t)\Phi\, {}^{Q}a_{D^{o}}(\varphi) + T^{o}(t)(\varphi - \Phi\, {}^{Q}a_{D^{o}}(\varphi)).$$

Since $D^{o}(\varphi - \Phi D^{o}(\varphi)) = 0$ and each column of Φ is in $\mathscr{D}(A^{o})$, the definition of $a_{D^{o}}$ and Lemmas 4.1 and 3.2 imply the existence of $K = K(a)$ such that

$$|T^{o}(t)\varphi| \le Ke^{at}|D^{o}(\varphi)| + Ke^{at}|\varphi - \Phi\, {}^{Q}D^{o}(\varphi)|.$$

Since D is continuous, this completes the proof of the theorem for the case $\alpha \equiv 0$, $L \equiv 0$.

Relation (4.7) follows as in the proof of Theorem 3.1 in [9].

<u>Case 2</u>. $\alpha \not\equiv 0$, $L \not\equiv 0$. In this case, $\Delta(\lambda)$ is given by (2.5),

$$\begin{aligned} \det \Delta(\lambda) &= \lambda^{n} \det \Delta^{o}(\lambda) + h(\lambda) \\ \operatorname{adj} \Delta(\lambda) &= \lambda^{n-1} \operatorname{adj} \Delta^{o}(\lambda) + G(\lambda) \\ \Delta(\lambda)^{-1} &= [\det \Delta(\lambda)]^{-1} \operatorname{adj} \Delta(\lambda) \\ &= \frac{1}{\lambda}\Delta^{o}(\lambda)^{-1} + W(\lambda) \\ W(\lambda) &= \frac{G(\lambda) - \Delta(\lambda)^{-1}h(\lambda)}{\lambda^{n} \det \Delta^{o}(\lambda)} \end{aligned} \tag{4.9}$$

where $\operatorname{adj} \Delta(\lambda)$, $\operatorname{adj} \Delta^{o}(\lambda)$ designate the cofactor matrices of

$\Delta(\lambda)$, $\Delta^o(\lambda)$, respectively. If $a > a_{D_o}$, then Lemma 3.4, the facts that μ^o, η are of bounded variation, μ^o is nonatomic at zero and $\alpha \in \mathscr{L}^1([-r,0],E^{n^2})$ imply that $h(\lambda) = \mathscr{O}(\lambda^{n-1})$, $G(\lambda) = \mathscr{O}(\lambda^{n-2})$, $W(\lambda) = \mathscr{O}(\lambda^{-2})$ as $|\lambda| \to \infty$, $\mathrm{Re}\ \lambda = a$.

Using standard Laplace transform techniques, for any φ in $\mathscr{D}(A) \cap Q$, $T(t)\varphi$ is given by

$$(4.10) \qquad T(t)\varphi(\theta) = \int_{C_a} e^{\lambda t}[\Delta^{-1}(\lambda)e^{\lambda\theta}(D(\varphi) - \lambda \int_{-r}^{0} d\mu(\beta)\int_{0}^{\beta} e^{\lambda(\beta-\alpha)}\varphi(\alpha)d\alpha$$
$$- \int_{-r}^{0} d\eta(\beta)\int_{0}^{\beta} e^{\lambda(\beta-\alpha)}\varphi(\alpha)d\alpha) - \int_{0}^{\theta} e^{\lambda(\theta-\alpha)}\varphi(\alpha)d\alpha]d\lambda$$

where μ is the $n \times n$ matrix function of bounded variation given by $\mu(\theta) = \mu^o(\theta) + \int_0^\theta \alpha(s)ds$ and

$$\int_{C_a} = (2\pi i)^{-1}\lim_{\omega \to \infty} \int_{a-i\omega}^{a+i\omega} .$$

The term containing $\int_0^\theta e^{\lambda(\theta-\alpha)}\varphi(\alpha)d\alpha$ and the one containing η are treated in the same manner as in the proof of Theorem IV.1 of [5]. Using the fact that $\Delta^{-1}(\lambda)$ is given by (4.9), the remaining terms in (4.10) may be written as

$$\int_{C_a} e^{\lambda t}[\frac{1}{\lambda}\Delta^o(\lambda)^{-1} + W(\lambda)]e^{\lambda\theta}[D(\varphi) - \lambda \int_{-r}^{0} d\mu(\beta) \int_{0}^{\beta} e^{\lambda(\beta-\alpha)}\varphi(\alpha)d\alpha]d\lambda$$
$$= T^o(t)\varphi + \int_{C_a} e^{\lambda t}W(\lambda)e^{\lambda\theta}[D(\varphi) - \lambda \int_{-r}^{0} d\mu(\beta) \int_{0}^{\beta} e^{\lambda(\beta-\alpha)}\varphi(\alpha)d\alpha]d\lambda.$$

The first term in this expression was treated in case 1. Since

$W(\lambda) = \mathcal{O}(\lambda^{-2})$ as $|\lambda| \to \infty$, Re $\lambda = a$, the first term in the integral admits an estimate of the form $Ke^{at}|\varphi|$. Since $\lambda W(\lambda) = \mathcal{O}(\lambda^{-1})$ as $|\lambda| \to \infty$, Re $\lambda = a$, the last term in the integral can be shown to have an estimate of the same form by using arguments similar to the one used for the η terms above.

Since $\mathcal{D}(A)$ is dense in C, estimate (4.6) holds for all φ in $C \cap Q_a$. Relation (4.7) is verified as in the proof of Theorem 3.1 in [8]. This completes the proof.

Corollary 4.1. Suppose D is given in (4.1), $a_{D^0} < 0$, and all roots of (2.5) have negative real parts. Then there is an $\alpha > 0$, $K > 0$ such that

$$|T(t)\varphi| \leq Ke^{-\alpha t}|\varphi|, \quad t \geq 0, \quad \varphi \in C,$$
$$|B_t| + \int_0^1 |d_s B_{t-s}| \leq Ke^{-\alpha t}, \quad t \geq 0.$$

Proof: Use Theorem 4.1 with $a = -\alpha$ greater than all roots of (2.5).

5. The saddle point property. Suppose D, L satisfy (2.1). In this section, we consider the linear system (2.2) along with the perturbed linear system

$$(5.1) \qquad \frac{d}{dt}[D(x_t) - G(x_t)] = L(x_t) + F(x_t)$$

where F, G satisfy the relations

$$\text{(5.2)} \qquad \begin{aligned} &F(0) = 0, \quad G(0) = 0 \\ &|F(\varphi) - F(\psi)| \leq \mu(\sigma)|\varphi - \psi| \\ &|G(\varphi) - G(\psi)| \leq \mu(\sigma)|\varphi - \psi| \end{aligned}$$

for $|\varphi|$, $|\psi| < \sigma$ and some continuous nondecreasing function $\mu(\sigma)$ with $\mu(0) = 0$.

It will also be assumed that the roots of the characteristic equation

$$\text{(5.3)} \qquad \det \Delta(\lambda) = 0, \ \Delta(\lambda) = \lambda[I - \int_{-r}^{0} e^{\lambda\theta} d\mu(\theta)] - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta)$$

have nonzero real parts and $a_D < 0$, where a_D is defined in Definition 3.1. This latter assumption implies that the space C can be decomposed as

$$C = U \oplus S,$$

where U is finite dimensional and the semigroup $T(t)$ generated by (2.2) can be defined on U for all $t \in (-\infty, \infty)$ and there are $K > 0$, $\alpha > 0$ such that

$$\text{(5.4)} \qquad \begin{aligned} &|T(t)\varphi| \leq Ke^{\alpha t}|\varphi|, \qquad t \leq 0, \quad \varphi \in U \\ &|T(t)\varphi| \leq Ke^{-\alpha t}|\varphi|, \qquad t \geq 0, \quad \varphi \in S. \end{aligned}$$

For any $\varphi \in C$, we write $\varphi = \varphi^U + \varphi^S$, $\varphi^U \in U$, $\varphi^S \in S$. The decomposition of C as $U \oplus S$ defines two projection operators $\pi_U: C \to U$, $\pi_U U = U$, $\pi_S: C \to S$, $\pi_S S = S$, $\pi_S = I - \pi_U$.

Suppose K, α are defined in (5.4) and $x(\varphi)$ is the solution of (5.1) with initial value φ at zero. For any $\delta > 0$, let $B_\delta = \{\varphi \in C: |\varphi| \leq \delta\}$ and

$$\mathcal{S}_\delta = \{\varphi \in C: \varphi^S \in B_{\delta/2K},\ x_t(\varphi) \in B_\delta,\ t \geq 0\}, \tag{5.5}$$

$$\mathcal{U}_\delta = \{\varphi \in C: \varphi^U \in B_{\delta/2K},\ x_t(\varphi) \in B_\delta,\ t \leq 0\}.$$

If Γ is a subset of C which contains zero, we say Γ <u>is tangent to</u> S <u>at zero</u> if $|\pi_U\varphi|/|\pi_S\varphi| \to 0$ as $\varphi \to 0$ in Γ. Similarly, Γ <u>is tangent to</u> U <u>at zero</u> if $|\pi_S\varphi|/|\pi_U\varphi| \to 0$ as $\varphi \to 0$ in Γ.

We now give the main result of this section, generalizing a theorem of Hale and Perelló [10] for retarded functional differential equations.

<u>Theorem 5.1.</u> With the notation as above, there is a $\delta > 0$ such that π_S is a homeomorphism from the set $\mathcal{S}_\delta$ onto $S \cap B_{\delta/2K}$ and $\mathcal{S}_\delta$ is tangent to S at zero. Also, π_U is a homeomorphism from the set $\mathcal{U}_\delta$ onto $U \cap B_{\delta/2K}$ and $\mathcal{U}_\delta$ is tangent to U at zero. Furthermore, there are positive constants M, γ such that

$$|x_t(\varphi)| \leq Me^{-\gamma t}|\varphi|,\quad t \geq 0,\quad \varphi \text{ in } \mathcal{S}_\delta, \tag{5.6}$$

$$|x_t(\varphi)| \leq Me^{\gamma t}|\varphi|,\quad t \leq 0,\quad \varphi \text{ in } \mathcal{U}_\delta.$$

Finally, if $F(\varphi)$, $G(\varphi)$ have continuous Frechét derivatives with respect to φ and $h_S: S \cap B_{\delta/2K} \to \mathcal{S}_\delta$, $h_U: U \cap B_{\delta/2K} \to \mathcal{U}_\delta$ are defined by $h_S\varphi = \pi_S^{-1}\varphi$, $\varphi \in S \cap B_{\delta/2K}$, $h_U\varphi = \pi_U^{-1}\varphi$, $\varphi \in U \cap B_{\delta/2K}$, then h_S and h_U have continuous Frechét derivatives.

Proof: The proof will follow as much as possible the proof of the saddle point property for ordinary differential equations given in Hale [11]. Using the above decomposition of C, the solution $x = x(\varphi)$ of (5.1) can be written as

$$\text{(5.7)} \qquad \begin{aligned} &\text{(a)}\quad x_t = x_t^S + x_t^U \\ &\text{(b)}\quad x_t^S = T(t-\sigma)x_\sigma^S + \int_\sigma^t B_{t-s}^S[d_s G(x_s) + F(x_s)ds] \\ &\text{(c)}\quad x_t^U = T(t-\sigma)x_\sigma^U + \int_\sigma^t B_{t-s}^U[d_s G(x_s) + F(x_s)ds] \end{aligned}$$

for any $\sigma \in (-\infty,\infty)$. Furthermore, K,α can be chosen so that

$$\text{(5.8)} \qquad \begin{aligned} &|B_t^U| + \int_{-1}^0 |d_s B_{t-s}^U| \le Ke^{\alpha t}, \quad t \le 0 \\ &|B_t^S| + \int_0^1 |d_s B_{t-s}^S| \le Ke^{-\alpha t}, \quad t \ge 0. \end{aligned}$$

Relations (5.8) also imply that K can be chosen so that

$$\text{(5.9)} \qquad \begin{aligned} &\text{(a)}\quad \int_\tau^0 |d_s B_{t-s}^U| \le Ke^{\alpha(t-\tau)}, \quad t \le \tau \le 0 \\ &\text{(b)}\quad \int_0^\tau |d_s B_{t-s}^S| \le Ke^{-\alpha(t-\tau)}, \quad t \ge \tau \ge 0. \end{aligned}$$

Using relation (5.9) and proceeding in a manner very similar to [10], one finds that for any solution of (5.1) which exists and is bounded for $t \geq 0$, there is a φ^S in S such that

$$x_t = T(t)\varphi^S + \int_0^t B^S_{t-s}[d_s G(x_s) + F(x_s)ds] + \int_\infty^0 B^U_{-s}[d_s G(x_{t+s}) + F(x_{t+s})ds] \tag{5.10}$$

for $t \geq 0$. Also, for any solution x of (5.1) which exists and is bounded for $t \leq 0$, there is a φ^U in U such that

$$x_t = T(t)\varphi^U + \int_0^t B^U_{t-s}[d_s G(x_s) + F(x_s)ds] + \int_{-\infty}^0 B^S_{-s}[d_s G(x_{t+s}) + F(x_{t+s})ds] \tag{5.11}$$

for $t \leq 0$. Conversely, any solution of (5.10) bounded on $[0,\infty)$ and any solution of (5.11) bounded on $(-\infty,0]$ is a solution of (5.1). Of course, estimates made in the integrals involving G are made using the relation

$$\int_\sigma^t B_{t-s} d_s G(x_s) = -B_{t-\sigma}G(x_\sigma) - \int_\sigma^t [d_s B_{t-s}]G(x_s). \tag{5.12}$$

We first discuss the solution of (5.10) for any φ^S sufficiently small. Suppose K, α are the constants used in (5.6), (5.8), (5.9) and $\mu(\sigma)$, $\sigma \geq 0$, is the function given in (5.2).

Choose $\delta > 0$ so small that

$$(5.13)\qquad (8K+4K/\alpha)\mu(\delta) < 1, \quad 8K^2(1+\alpha^{-1})(1+\mu(\delta))\mu(\delta) < 1/2$$

and define $\mathcal{G}(\delta)$ as the set of continuous functions $y: [0,\infty) \to C$ such that $|y| \overset{\text{def}}{=} \sup_{0\leq t<\infty}|y_t| \leq \delta/2$, $y_0^S = 0$. The set $\mathcal{G}(\delta)$ is a closed bounded subset of the Banach space $C([0,\infty),C)$ of all bounded continuous functions mapping $[0,\infty)$ into C with the uniform topology. For any y in $\mathcal{G}(\delta)$ and any φ^S in S, $|\varphi^S| \leq \delta/2K$, define the transformation $\mathcal{P} = \mathcal{P}(\varphi^S)$ taking $\mathcal{G}(\delta)$ into $C([0,\infty),C)$ by

$$(5.14)\qquad (\mathcal{P}y)_t = \int_0^t B_{t-s}^S[d_sG(y_s+T(s)\varphi^S) + F(y_s+T(s)\varphi^S)ds]$$

$$+ \int_\infty^0 B_{-s}^U[d_sG(y_{t+s}+T(t+s)\varphi^S) + F(y_{t+s}+T(t+s)\varphi^S)ds]$$

for $t \geq 0$. It is easy to see that $\mathcal{P}y \in C([0,\infty),C)$ and $(\mathcal{P}y)_0^S = 0$. Also, $|y_t + T(t)\varphi^S| \leq \delta$ for all $t \geq 0$. Consequently, from (5.12), (5.13), (5.14), (5.4) and (5.8),

$$|(\mathcal{P}y)_t| \leq (4K + \frac{2K}{\alpha})\mu(\delta)\delta < \delta/2$$

and $\mathcal{P}: \mathcal{G}(\delta) \to \mathcal{G}(\delta)$. Furthermore, in a similar manner,

$$|(\mathcal{P}y)_t - (\mathcal{P}z)_t| \leq (4K + \frac{2K}{\alpha})\mu(\delta)|y - z| \leq \frac{1}{2}|y - z|$$

for $t \geq 0$, $y, z \in \mathscr{G}(\delta)$ and $\mathscr{P}$ is a uniform contraction on $\mathscr{G}(\delta)$. Thus, $\mathscr{P}$ has a unique fixed point $y^* = y^*(\varphi^S)$ in $\mathscr{G}(\delta)$. The function $x_t^* = y_t^* + T(t)\varphi^S$ obviously satisfies (5.10) and is the unique solution of (5.10) with $|y_t| \leq \delta/2$ and $x_0^S = \varphi^S$. The fact that $\mathscr{P}$ is a uniform contraction on $\mathscr{G}(\delta)$ implies that $y^*(\varphi^S)$ and therefore $x^*(\varphi^S)$ are continuous in φ^S.

With x^* defined as above, if $x^* = x^*(\varphi^S)$, $\tilde{x}^* = x^*(\tilde{\varphi}^S)$, then

$$x_t^* - \tilde{x}_t^* = T(t)(\varphi^S - \tilde{\varphi}^S) - B_t^S[G(\varphi^S) - G(\tilde{\varphi}^S)]$$

$$- \int_0^t [d_s B_{t-s}^S][G(x_s^*) - G(\tilde{x}_s^*)] + \int_0^t B_{t-s}^S[F(x_s^*) - F(\tilde{x}_s^*)]ds$$

$$- \int_\infty^0 [d_s B_{-s}^U][G(x_{t+s}^*) - G(\tilde{x}_{t+s}^*)] + \int_\infty^0 B_{-s}^U[F(x_{t+s}^*) - F(\tilde{x}_{t+s}^*)]ds.$$

Consequently, if $u(t) = |x_t^* - \tilde{x}_t^*|$, $\mu = \mu(\delta)$, then (5.4) and (5.9) imply that

$$(5.15) \quad u(t) \leq K(1+\mu)e^{-\alpha t}u(0)$$

$$+ \mu \int_0^t |d_s B_{t-s}^S| u(s) + K\mu \int_0^t e^{-\alpha(t-s)}u(s)ds$$

$$+ \mu \int_0^\infty |d_s B_{-s}^U| u(t+s) + K\mu \int_0^\infty e^{-\alpha s}u(t+s)ds.$$

For any $t \geq \tau \geq 0$, relation (5.8), (5.9) and this latter expression with the first integral written as $\int_0^t = \int_0^\tau + \int_\tau^t$ imply that

$$(5.16)\quad u(t) \le K(1+\mu)e^{-\alpha t}u(0) + K\mu e^{-\alpha(t-\tau)}\sup_{0\le s\le \tau} u(s)$$

$$+ K\mu \sup_{\tau\le s\le t} u(s) + K\mu \int_0^t e^{-\alpha(t-s)}u(s)ds$$

$$+ K\mu \sup_{0\le s} u(t+s) + K\mu \int_0^\infty e^{-\alpha s}u(t+s)ds.$$

We first show that $u(t) \to 0$ as $t \to \infty$. If this is not the case, $u(t)$ bounded for $t \ge 0$ implies there is a $\nu > 0$ such that $\overline{\lim}_{t\to\infty} u(t) = \nu > 0$. For any $0 < \theta < 1$, there is a $t_1 > 0$ such that $u(t) \le \theta^{-1}\nu$, $t \ge t_1$. Consequently, for $\tau = t_1$ in (5.16) and $t \ge t_1$, this yields

$$u(t) \le K(1+\mu)e^{-\alpha t}u(0) + K\mu e^{-\alpha(t-t_1)}\sup_{0\le s\le t_1} u(s)$$

$$+ K\mu\theta^{-1}\nu + K\mu \int_0^{t_1} e^{-\alpha(t-s)}u(s)ds + \frac{K\mu}{\alpha}\theta^{-1}\nu$$

$$+ K(1+\frac{1}{\alpha})\mu\theta^{-1}\nu.$$

The right hand side of this equation has a limit as $t \to \infty$ which is

$$2K(1+\frac{1}{\alpha})\mu\theta^{-1}\nu < \frac{1}{2}\theta^{-1}\nu < \theta^{-1}\nu.$$

Therefore, $\overline{\lim}_{t\to\infty} u(t) < \nu$ which is a contradiction. Thus, $u(t) \to 0$ as $t \to \infty$.

Since $u(t) \to 0$ as $t \to \infty$, $u(t)$ has a maximum and an argument similar to the preceeding shows that $u(t) = 0$ if $u(0) = 0$.

Thus there will be a constant such that $u(t) \leq (\text{const})\, u(0)$, $t \geq 0$.

Let $v(t) = \sup_{t \leq s} u(s)$. Since $u(t) \to 0$ as $t \to \infty$, for every $t \geq 0$, there is a $t_1 \geq t$ such that $v(t) = v(s) = u(t_1)$ for $t \leq s \leq t_1$, $v(s) < v(t_1)$ for $s > t_1$. Therefore, from (5.15),

$$\begin{aligned}
v(t) = u(t_1) &\leq K(1+\mu)e^{-\alpha t_1}u(0) + K\mu \sup_{0\leq s\leq t_1} u(s) \\
&\quad + K\mu\left(\int_0^t + \int_t^{t_1}\right)e^{-\alpha(t_1-s)}v(s)ds + K\mu(1+\tfrac{1}{\alpha})v(t_1) \\
&\leq K(1+\mu)e^{-\alpha t}v(0) + K\mu \sup_{0\leq s\leq t_1} v(s) \\
&\quad + K\mu \int_0^t e^{-\alpha(t_1-s)}v(s)ds + K\mu(1+\tfrac{2}{\alpha})v(t) \\
&\leq K(1+\mu)e^{-\alpha t}v(0) + K\mu \sup_{0\leq s\leq t} v(s) \\
&\quad + K\mu \int_0^t e^{-\alpha(t-s)}v(s)ds + K\mu(1+\tfrac{2}{\alpha})v(t).
\end{aligned}$$

Since $K\mu(1+2\alpha^{-1}) < 1/2$, we have

$$(5.17) \qquad v(t) \leq K_1(\delta)e^{-\alpha t}v(0) + K_2(\delta)\sup_{0\leq s\leq t} v(s) + K_2(\delta)\int_0^t e^{-\alpha(t-s)}v(s)ds$$

where

$$K_1(\delta) = \frac{K(1+\mu(\delta))}{1-K\mu(\delta)(1+2\alpha^{-1})} < 2K(1+\mu(\delta)),$$

$$K_2(\delta) = \frac{K\mu(\delta)}{1-K\mu(\delta)(1+2\alpha^{-1})} < 2K\mu(\delta).$$

Our next objective is to show that $v(t)$ satisfying (5.17) approaches zero exponentially. To do this, we first show that

$$v(t) \leq K_3(\delta)v(0), \quad K_3(\delta) = 2K_1(\delta). \tag{5.18}$$

In fact, if this is not the case then there is a $\tau > 0$ such that $v(t) < K_3(\delta)v(0)$ for $0 < t < \tau$, $v(\tau) = K_3(\delta)v(0)$. Consequently, (5.17) implies

$$\begin{aligned} K_3(\delta)v(0) = v(\tau) &\leq [K_1(\delta) + K_2(\delta)K_3(\delta) + K_2(\delta)K_3(\delta)\alpha^{-1}]v(0) \\ &= [\tfrac{1}{2} + K_2(\delta)(1+\alpha^{-1})]K_3(\delta)v(0) \\ &< K_3(\delta)v(0) \end{aligned}$$

since $K_2(\delta)(1+\alpha^{-1}) < 2K\mu(\delta)(1+\alpha^{-1}) < 1/2$. This contradiction shows that (5.18) is satisfied for all $t \geq 0$.

Using (5.18) in (5.17), we have

$$v(t) \leq K_1(\delta)e^{-\alpha t}v(0) + K_2(\delta)K_3(\delta)(1+\alpha^{-1})v(0), \quad t \geq 0.$$

Choose $\beta > 0$ so that $K_1(\delta)e^{-\alpha\beta} < 1/4$. Since $K_2(\delta)K_3(\delta)(1+\alpha^{-1}) < 1/2$, it follows that $v(\beta) \leq (3/4)v(0)$. Finally, since the initial value 0 has no particular significance for autonomous equations, it follows that $v(t+\beta) \leq v(t)$ for all $t \geq 0$. This clearly implies the existence of an $\alpha_1 > 0$, $K_4 > 0$ such that

$$v(t) \leq K_4 e^{-\alpha_1 t} v(0).$$

Consequently, returning to the definition of v and u, we have

$$|x^*(\varphi^S) - x^*(\tilde{\varphi}^S)| \leq Me^{-\alpha_1 t}|\varphi^S - \tilde{\varphi}^S|, \quad t \geq 0.$$

Since $x^*(0) = 0$, this implies (5.6) is satisfied.

The above argument has also shown that

$$\mathcal{S}_\delta = \{\varphi \in C: \varphi = x_0^*(\varphi^S), \varphi^S \text{ in } S, |\varphi^S| \leq \delta/2K\}.$$

If $h_S: S \cap B_{\delta/2K} \to \mathcal{S}_\delta$ is defined by $h_S\varphi^S = x_0^*(\varphi^S)$, then h_S is continuous and

$$h_S(\varphi^S) = \varphi^S + \int_\infty^0 B_{-s}^U[d_s G(x_s^*(\varphi^S)) + F(x_s^*(\varphi^S))ds].$$

Also, with an argument similar to the above, one shows that $|h_S(\varphi^S) - h_S(\tilde{\varphi}^S)| \geq |\varphi^S - \tilde{\varphi}^S|/2$ for all φ^S, $\tilde{\varphi}^S$ in $S \cap B_{\delta/2K}$, and thus, h_S is one-to-one. Since $h_S^{-1} = \pi_S$ is continuous, it follows that h_S is a homeomorphism.

From the fact that $x_0^*(0) = 0$, $x^*(\varphi^S)$ satisfies (5.6) and

$$\pi_U x_0^*(\varphi^S) = -\int_\infty^0 [d_s B_{-s}^U]G(x_s^*(\varphi^S)) + \int_\infty^0 B_{-s}^U F(x_s^*(\varphi^S))ds,$$

we also have

$$|\pi_U x_0^*(\varphi^S)| \leq 2K^2(1+\alpha^{-1})\mu(2K|\varphi^S|)|\varphi^S|$$

and this shows that $\mathscr{S}_\delta$ is tangent to S at zero.

If F, G have continuous Frechet derivatives $F'(\varphi)$, $G'(\varphi)$ and satisfy (5.2), then $|F'(\varphi)| \leq \mu(\delta)$ for $|\varphi| < \delta$. From (5.14), it follows that the derivative $\mathscr{P}'(y)$ of $\mathscr{P}y$ with respect to φ^S evaluated at ψ^S in S is

$$\begin{aligned}(\mathscr{P}'(y)\psi^S)_t = \int_0^t B^S_{t-s}[d_s G'(y_s+T(s)\varphi^S)T(s)\psi^S + F'(y_s+T(s)\varphi^S)T(s)\psi^S ds] \\ + \int_\infty^0 B^U_{-s}[d_s G'(y_s+T(s)\varphi^S)T(s)\psi^S + F'(y_{t+s}+T(t+s)\varphi^S)T(t+s)\psi^S ds], \\ t \geq 0.\end{aligned}$$

Since $|T(s)\psi^S| \leq K|\psi^S|$ and $\mu(\delta)$ satisfies (5.13), it follows that

$$|(\mathscr{P}'(y)\psi^S)_t| \leq K^2(1+\alpha^{-1})\mu(\delta)|\psi^S| < \frac{1}{16}|\psi^S|, \qquad t \geq 0.$$

Using the fact that the mapping $\mathscr{P}$ is a uniform contraction on $\mathscr{S}(\delta)$, one obtains the differentiability of $h_S(\varphi^S)$ with respect to φ^S. The argument for $\mathscr{U}_\delta$ is applied similarly to the above to complete the proof of Theorem 5.1.

<u>Corollary 5.1.</u> Under the hypothesis of Theorem 5.1, there is a $\delta > 0$ such that each solution of (5.1) with initial value in B_δ either approaches zero as $t \to \infty$ (and then exponentially) or leaves

B_δ for some finite time. Any solution with initial value in B_δ which is defined for $t \leq -r$ must either approach zero as $t \to -\infty$ or leave B_δ for some finite negative time.

Proof: There is a $k \geq 1$ such that $|\varphi^S| \leq k|\varphi|$ for all φ in C. Suppose δ is given as in Theorem 5.1 and choose $0 < \delta \leq \delta/2Kk$. This δ_1 serves for the δ of the corollary. A similar argument applies to the last statement of the corollary.

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